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## ON ORDERED MONOID RINGS

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A monoid  $M$  is said to be *ordered* if the elements of  $M$  are linearly ordered with respect to the relation  $<$  and that, for all  $x, y, z \in G$ ,  $x < y$  implies  $zx < zy$  and  $xz < yz$ . It is well known that torsion-free nilpotent groups and free groups are ordered groups (see [8, Lemma 13.1.6 and Corollary 13.2.8]). Hence any submonoid of a torsion-free nilpotent group or a free group is an ordered monoid.

Let  $R$  be a ring. A left (resp. right) annihilator of a subset  $U$  of  $R$  is defined by  $l_R(U) = \{a \in R \mid aU = 0\}$  (resp.  $r_R(U) = \{a \in R \mid Ua = 0\}$ ). Let  $G$  be an ordered monoid. Put  $rAnn_R(2^R) = \{r_R(U) \mid U \subseteq R\}$  and  $lAnn_R(2^R) = \{l_R(U) \mid U \subseteq R\}$ . If  $U$  is a subset of  $R$ , then  $r_{RG}(U) = r_R(U)RG$ . Hence we have a map  $\Phi : rAnn_R(2^R) \longrightarrow rAnn_{RG}(2^{RG})$  defined by  $\Phi(I) = I(RG)$  for every  $I \in rAnn(R)$ . For an element  $f \in RG$ ,  $C_f$  denotes the set of coefficients of  $f$  and for a subset  $V$  of  $RG$ ,  $C_V$  denotes the set  $\cup_{f \in V} C_f$ . Then  $r_{RG}(V) \cap R = r_R(V) = r_R(C_V)$ . Hence we also have a map  $\Psi : rAnn_{RG}(2^{RG}) \longrightarrow rAnn_R(2^R)$  defined by  $\Psi(I) = I \cap R$  for every  $I \in \Delta$ . Obviously  $\Phi$  is injective and  $\Psi$  is surjective. Clearly  $\Phi$  is surjective if

and only if  $\Psi$  is injective, and in this case  $\Phi$  and  $\Psi$  are the inverses of each other. We consider when  $\Phi$  is surjective.

Following Rege and Chhawchharia [10] a ring  $R$  is called an *Armendariz ring* if whenever polynomials  $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ , satisfy  $f(x)g(x) = 0$  we have  $a_i b_j = 0$  for every  $i$  and  $j$ . This name is conneted with the work of Armendariz [2]. Some results of Armendariz rings can be found in [1], [5], [7] and [10]. Let  $G$  be an ordered monoid. A ring  $R$  is called a *G-Armendariz ring* if whenever  $p = \sum_{g \in G} a_g g, q = \sum_{h \in G} b_h h \in RG$  satisfy  $pq = 0$  we have  $a_g b_h = 0$  for every  $g$  and  $h$  in  $G$ .

The following proposition shows that  $\Phi$  is bijective if and only if  $R$  is Armendariz.

**Proposition 1.** Let  $R$  be a ring and let  $G$  be an ordered monoid. The following statements are equivalent:

- 1)  $R$  is  $G$ -Armendariz.
- 2)  $rAnn_R(2^R) \longrightarrow rAnn_{RG}(2^{RG}); A \rightarrow A(RG)$  is bijective.
- 3)  $lAnn_R(2^R) \longrightarrow lAnn_{RG}(2^{RG}); B \rightarrow (RG)B$  is bijective.

Following Kaplansky [6], a ring  $R$  is called a *Baer ring* if the left annihilator of each subset is generated by an idempotent. We note that the definition of Baer rings is left-right symmetric. A ring  $R$  is called a *left (resp. right) p.p.-ring* if the left (resp. right) annihilator of each element of  $R$  is generated by an idempotent. A left and right p.p. ring is celled a p.p. ring. Using Proposition 1, we can generalize [7, Theorems 9 and 10]) as follows:

**Corollary 2.** Let  $G$  be an ordered monoid and let  $R$  be a  $G$ -Armendariz ring. Then  $R$  is a Baer ring (resp. p.p. ring) if and only if  $RG$  is a Baer ring (resp. p.p. ring).

A ring  $R$  is called a *G-quasi-Armendariz ring* if whenever  $p = \sum_{i=0}^m a_i g_i$ ,  $q = \sum_{j=0}^n b_j h_j \in RG$  satisfy  $pRGq = 0$ , then we have  $a_i R b_j = 0$  for every  $i$  and  $j$ . In case  $G = \{x^i \mid i = 0, 1, 2, \dots\}$ , a *G-quasi-Armendariz ring* is simply called a *quasi-Armendariz ring*. In [5], we studied quasi-Armendariz rings. Let  $rAnn_R(id(R))$  (resp.  $lAnn_R(id(R))$ ) denote the set  $\{r_R(U) \mid U \text{ is an ideal of } R\}$  (resp.  $\{l_R(U) \mid U \text{ is an ideal of } R\}$ ).

**Proposition 3.** Let  $R$  be a ring and let  $G$  be an ordered monoid. Then the following statements are equivalent:

- 1)  $R$  is *G-quasi-Armendariz*.
- 2)  $rAnn_R(id(R)) \longrightarrow rAnn_{RG}(id(RG)); A \rightarrow A(RG)$  is bijective.
- 3)  $lAnn_R(id(R)) \longrightarrow lAnn_{RG}(id(RG)); B \rightarrow (RG)B$  is bijective.

A submodule  $N$  of a left  $R$ -module  $M$  is called a *pure submodule* if  $L \otimes_R N \rightarrow L \otimes_R M$  is a monomorphism for every right  $R$ -module  $L$ .

**Theorem 4.** Let  $G$  be an ordered monoid. Then the following are equivalent;

- (1)  $l_R(Ra)$  is pure as a left ideal in  $R$  for any element  $a \in R$ ;
- (1)  $l_{RG}(RGz)$  is pure as a left ideal in  $RG$  for any element  $z \in RG$ ;

In this case,  $R$  is a *G-quasi-Armendariz ring*.

**Corollary 5.** Let  $R$  be a commutative ring and let  $G$  be an abelian ordered monoid. Then each principal ideal of  $R$  is flat if and only if each principal ideal of  $RG$  is flat. In this case  $R$  is a *G-Armendariz ring*.

A ring  $R$  is called *quasi-Baer* if the left annihilator of every left ideal of  $R$  is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [3] and [9].

Let  $R$  be a quasi-Baer ring and let  $a \in R$ . Then  $l_R(aR) = Re$  for some idempotent  $e \in R$ , and so  $l_R(aR)$  is pure as a left ideal in  $R$ . Therefore a quasi-Baer ring satisfies the hypothesis of Theorem 4. Hence we obtain the following.

**Corollary 6** ([4, Theorem 1]). Let  $G$  be an ordered monoid. A ring  $R$  is a quasi-Baer ring if and only if  $RG$  is a quasi-Baer ring. In this case,  $R$  is a quasi-Armendariz ring.

Now we consider some extensions of  $G$ -quasi-Armendariz rings. Let  $R$  be a ring and let  $n$  be a positive integer. Let  $M_n(R)$  denote the ring of  $n \times n$  matrices over  $R$  and  $e_{ij}$  denote the  $(i, j)$ -matrix unit.  $T_n(R)$  denotes the ring of all  $n \times n$  upper triangular matrices over  $R$ .

**Theorem 7.** Let  $G$  be an ordered monoid. If  $R$  a  $G$ -quasi-Armendariz ring and let  $S$  be a subring of  $M_n(R)$  such that  $e_{ii}Se_{jj} \subseteq S$  for all  $i, j \in \{1, \dots, n\}$ . Then  $S$  is also a  $G$ -quasi-Armendariz ring.

**Corollary 8.** Let  $G$  be an ordered monoid. If  $R$  a  $G$ -quasi-Armendariz ring, then, for any positive integer  $n$ ,  $T_n(R)$  is also a  $G$ -quasi-Armendariz ring.

For  $f \in RG$ , the content  $A_f$  of  $f$  is the ideal of  $R$  generated by the coefficients of  $f$ . For any subset  $S$  of  $RG$ ,  $A_S$  denotes the ideal  $\sum_{f \in S} A_f$ . In case  $G = \{x^i \mid i = 0, 1, 2, \dots\}$ , a commutative ring  $R$  is *Gaussian* if  $A_{fg} = A_f A_g$  for all  $f, g \in R[x]$  (See [1]). We extend this notion to noncommutative rings as follows. A ring  $R$  is said to be *G-quasi-Gaussian* if  $A_{fRg} = A_f A_g$  for all  $f, g \in RG$ .

**Theorem 9.** A ring  $R$  is  $G$ -quasi-Gaussian if and only if every homomorphic

image of  $R$  is quasi-Armendariz.

**Example 10.** A ring  $R$  is *fully idempotent* if  $I^2 = I$  for every twosided ideal  $I$  of  $R$ . Obviously a ring  $R$  is fully idempotent if and only if every homomorphic image of  $R$  is semiprime. Von Neumann regular rings are fully idempotent. We can easily see that a semiprime ring is  $G$ -quasi-Armendariz. Therefore by Theorem 9, a fully idempotent ring is a  $G$ -quasi-Gaussian ring.

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